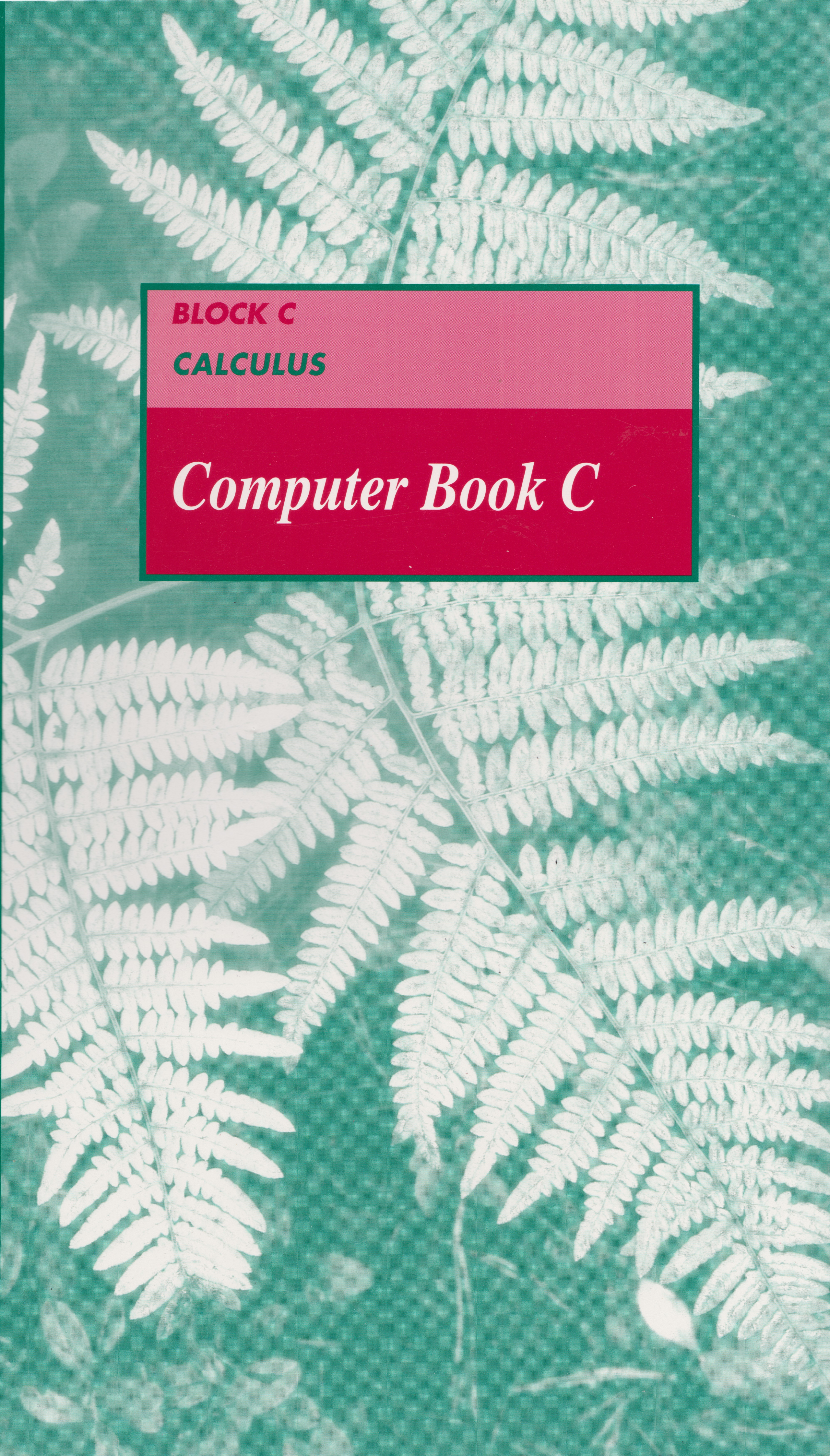
MS221 CB C

The Open University

A second level interdisciplinary course

Exploring Mathematics



BLOCK C

CALCULUS

Computer Book C

COMPUTER BOOK

C

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Prepared by the course team

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About this course

This computer book forms part of the course MS221 *Exploring Mathematics*. This course and the courses MU120 *Open Mathematics* and MST121 *Using Mathematics* provide a flexible means of entry to university-level mathematics. Further details may be obtained from the address below.

MS221 uses the software package Mathcad (MathSoft, Inc.) to investigate mathematical concepts and as a tool in problem solving. This software is provided as part of the course.

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Contents

Guidance notes

This computer book contains those sections of the chapters in Block C which require you to use Mathcad. Each of these chapters contains instructions as to when you should first refer to particular material in this computer book, so you are advised not to work on the activities here until you have reached the appropriate points in the chapters.

In order to use this computer book, you will need the following Mathcad files.

Chapter C1

221C1-01 Differentiation
221C1-02 A differentiation template
221C1-03 The Newton–Raphson method

Chapter C2

221C2-01 Integration templates
221C2-02 Volumes of solids of revolution (Optional)

Chapter C3

221C3-01 Taylor series and polynomials

Instructions for installing these files onto your computer's hard disk, and for opening them, are given in Chapter A0 of MST121.

The computer activities for Chapter C2 also require you to work with Mathcad worksheets which you have created yourself.

Activities based on software vary both in nature and in length. Sometimes the instructions for an activity appear only in the computer book; in other cases, instructions are given in the computer book and on screen. Feedback on an activity is sometimes provided on screen and sometimes given in the computer book.

For advice on how each computer session fits into suggested study patterns, refer to the Study guides in the chapters.

Chapter C1, Section 5
Differentiation with the computer

Mathcad can be used to find the derivatives of functions, which is the topic of Subsection 5.1. In Subsection 5.2, you will see the Newton–Raphson method implemented on the computer.

5.1 Finding derivatives

In Mathcad, derivatives are found using the $\frac{d}{dx}$ operator. This operator can be used both to find a general formula for the derivative of a function f, and to find the numerical value of the derivative of f at a particular point. Both these usages are introduced in this subsection.

Activities 5.1 and 5.2 below repeat activities in the course MST121 in which the Mathcad approach to finding derivatives is introduced, and their associated Mathcad file is identical to the corresponding file used in MST121. If you have recently studied Chapter C1 of MST121, then you should not find it necessary to work through Activities 5.1 and 5.2 in detail, though you may wish to refresh your memory. This also applies to the comments on each activity. If you have not recently studied Block C of MST121, then you should work through Activities 5.1 and 5.2 in the normal way.

Activity 5.1 Finding a formula for the derivative

Open Mathcad file **221C1-01 Differentiation**. Page 1 introduces the worksheet. Work through page 2, and then carry out Task 1 on page 3.

Remember to create your own working copy of the file.

Solutions are given on page 29.

Comment

◇ Make sure that the variables entered in the two placeholders of the $\frac{d}{dx}$ operator match. For example, if you mistakenly try to evaluate symbolically the expression
$$\frac{d}{dt}\cos(4x),$$
then you will obtain the answer 0.

◇ Brackets can be crucial when entering an expression into the right-hand placeholder of the $\frac{d}{dx}$ operator. For example, if the expression $x^3 - 6x^2 - 15x + 54$ is entered without being enclosed in brackets, Mathcad will differentiate the x^3 term but not the others.

You may find that Mathcad will not allow you to enter into the placeholder a quotient whose numerator is a sum of terms (it depends on how you enter the quotient). This difficulty can be avoided by enclosing the quotient in brackets.

An example of such a quotient is
$$\frac{\sin(t) - t^2}{e^t}.$$

To avoid such problems, it is good practice to include outer brackets around expressions that you enter into the right-hand placeholder of the $\frac{d}{dx}$ operator. This also helps to make the expression to be differentiated clear on the screen.

These symbolic keywords were introduced in MST121 Chapter A0, file 121A0-05. See also *A Guide to Mathcad* for further details.

- ◇ Sometimes the expression for a derivative obtained by Mathcad can be 'improved' by simplifying it. In place of symbolic evaluation ($\rightarrow$), either of the symbolic keywords 'simplify' and 'factor' can be applied to obtain a derivative. The outcome from these may or may not be the same as that from using $\rightarrow$, but will sometimes be in a more convenient form. However, what Mathcad regards as 'simpler' may not necessarily seem so to a human observer!

Mathcad notes

Remember that Mathcad notes are *optional*.

- ◇ The expressions e^x and $\exp(x)$ are equivalent in Mathcad, but the latter form is always used in the output of symbolic calculations, irrespective of the form used for input. Similarly, the power $\frac{1}{2}$ appears in output rather than the square root sign.
- ◇ If Mathcad is unable to carry out a symbolic operation (there are many possible reasons for this), then it will reproduce the input expression unchanged, apart from possible notational changes of the type just noted.

Activity 5.1 showed how to use the $\frac{d}{dx}$ operator and symbolic evaluation to obtain an algebraic expression for the derivative. This replicates what you might do by hand, but Mathcad can also be used to differentiate functions that would be rather complicated to do by hand. We turn next to how Mathcad can be applied to find the numerical value for the derivative at a particular point.

Activity 5.2 Evaluating the derivative at a point

You should still be working with Mathcad file 221C1-01.

Turn to page 4 of the worksheet, and carry out Task 2.

Solutions are given further down page 4 of the worksheet.

Comment

- ◇ Note the comments made towards the bottom of page 4 of the worksheet. Once a value has been defined for the differentiation variable (x, say), then the $\frac{d}{dx}$ operator can be evaluated either symbolically ($\frac{d}{dx}(\ldots) \rightarrow$) or numerically ($\frac{d}{dx}(\ldots) =$) to find the numerical value of the derivative at that particular value of x.

  These two evaluation methods usually give the same numerical value, but Mathcad calculates the two results in different ways. For symbolic evaluation, Mathcad finds a general formula for the derivative and then evaluates this formula for the particular value of x, whereas for numerical evaluation, Mathcad uses a numerical algorithm to find an approximate decimal value.

Essentially, this involves a direct application of the definition of derivative, as given by equation (1.2) in Chapter C1. See also the second Mathcad note below.

- ◇ If the expression $\frac{d}{dx}f(x)$ is entered, where a definition for the function $f(x)$ is provided earlier in the worksheet, then Mathcad can find either a symbolic derivative for $f(x)$ or the numerical value of this derivative at any specified point, just as before.

Mathcad notes

- ◇ When you enter $\rightarrow$ to evaluate an expression symbolically, or $=$ to evaluate it numerically, it doesn't matter where on the expression the blue editing lines are. All that matters is that the expression is complete, with every placeholder filled in. These two evaluation methods also behave in a similar way if a change is made to the worksheet, above or to the left of a calculation. In automatic calculation mode (the default), the result of a calculation involving either $\rightarrow$ or $=$ is updated automatically, while in manual mode, you can press the [F9] key to update the result.

To change from automatic to manual calculation mode, or vice versa, click on **Automatic Calculation** in the **Math** menu.

- ◇ When evaluating a derivative numerically ($\frac{d}{dx}(\ldots) =$), you must define earlier in the worksheet the point at which the derivative is to be found; for example, $x := 3$. Mathcad then uses a numerical algorithm to obtain an approximation to the exact value of the derivative at that point, which is usually accurate to 7 or 8 significant figures. Very occasionally the method fails, in which case the derivative is highlighted in red. Clicking on this expression reveals the error message 'Can't converge to a solution.'.

Without this prior definition, x appears in red in the $\frac{d}{dx}$ operator, as an undefined variable.

Now close file 221C1-01.

Mathcad provides a useful means of checking differentiations that you have carried out by hand. However, an answer supplied by Mathcad can look quite different from an answer obtained by hand, even if you use Mathcad to simplify the expression. The next activity gives you practice in checking that answers supplied by Mathcad are equivalent to answers obtained by hand.

The Mathcad file associated with this activity provides a 'differentiation template' that you may also wish to use in other circumstances.

Activity 5.3 Further derivatives

Open Mathcad file **221C1-02 A differentiation template**. The worksheet consists of a single page, which can be used to differentiate any function $f(x)$ that you input. The page is set up with the function

$$f(x) = \frac{x^2+1}{2x-1}.$$

Note that the output from applying the symbolic keywords 'simplify' and 'factor' to the derivative of this function is different from that obtained by straightforward symbolic evaluation.

Note also, towards the bottom of the page, that the template can also be used to provide a numerical value for the derivative $f'(x)$ at any specified point in its domain.

While the facilities for obtaining numerical values are included in the template for completeness, they do not otherwise feature in this activity.

Each of parts (a)–(e) overleaf gives a function $f(x)$ that you were asked to differentiate in the main text, together with the answer for $f'(x)$ provided in the solutions. In each case, enter the formula for the function into the definition of $f(x)$ in the template, and note the results. If it is not clear that one of the answers supplied by Mathcad is equivalent to the stated expression for $f'(x)$, then copy the closest-looking Mathcad answer onto a piece of paper and carry out further algebraic manipulation by hand to confirm the equivalence.

Remember to enter each multiplication, and to enclose the arguments of trigonometric functions in brackets. For example, the expressions for $f(x)$ in parts (a) and (c) can be entered by typing `x*e^x^2` and `cos((x+4)*sec(x))`, respectively. The square root symbol required for part (b) can be obtained from a button on the 'Calculator' toolbar, or by typing \ (backslash).

See Activity 2.6(a).

(a) $f(x) = xe^{x^2}, \qquad f'(x) = (2x^2+1)e^{x^2}$

See Activity 2.6(b).

(b) $f(x) = \dfrac{\sin(\sqrt{x})}{\sqrt{x}}, \qquad f'(x) = \dfrac{\sqrt{x}\cos(\sqrt{x}) - \sin(\sqrt{x})}{2x^{3/2}}$

See Activity 2.6(c).

(c) $f(x) = \cos((x+4)\sec x),$

$f'(x) = -\sec x(1+(x+4)\tan x)\sin((x+4)\sec x)$

See Activity 2.4(a).

(d) $f(x) = x^4 e^x \sin x, \qquad f'(x) = x^3 e^x((4+x)\sin x + x\cos x)$

See Activity 2.4(b).

(e) $f(x) = \dfrac{x\tan x}{1+x^2}, \qquad f'(x) = \dfrac{x(1+x^2)\sec^2 x + (1-x^2)\tan x}{(1+x^2)^2}$

Solutions are given on page 29.

Mathcad notes

It is safest to enter all products explicitly in Mathcad, by clicking on the 'Multiplication' × button on the 'Calculator' toolbar or by typing `*`. In some situations, for example when using brackets, Mathcad will assume that you intended to enter a product, and will insert the multiplication for you. However, Mathcad will *not* help in this way if you enter `xe` rather than `x*e` in part (a). In this case, Mathcad assumes that you have entered a single variable name, 'xe', rather than the product of the variable x and the exponential constant e.

Now close file 221C1-02.

Sometimes you may want to use Mathcad to find a second-order derivative. One way to do this is by using the $\frac{d}{dx}$ operator twice. You can enter the $\frac{d}{dx}$ operator in your worksheet, then enter the $\frac{d}{dx}$ operator again into the right-hand placeholder, and then fill in all the placeholders appropriately. For example, Mathcad gives the result

$$\frac{d}{dx}\left(\frac{d}{dx}(x^3 - 6x^2 - 15x + 54)\right) \to 6x - 12.$$

The keyboard alternative is `[Ctrl]?` (given by the three keys `[Ctrl]`, `[Shift]` and `/`).

Another way to find a second-order derivative in Mathcad is to use the $\frac{d^n}{dx^n}$ operator, for which there is a button on the 'Calculus' toolbar. You should enter '2' in the bottom placeholder (which will cause a 2 to appear in the top placeholder as well), and fill in the other placeholders just as for the $\frac{d}{dx}$ operator. You can evaluate the resulting expression either symbolically or numerically, in the same way as for expressions involving the $\frac{d}{dx}$ operator. For example, Mathcad gives the result

$$\frac{d^2}{dx^2}(x^3 - 6x^2 - 15x + 54) \to 6x - 12.$$

5.2 The Newton–Raphson method

You saw in the main text that a solution of the equation $f(x) = 0$ is called a 'zero' of the function f, and that it is often possible to find an approximation to a zero of a smooth function f by using the Newton–Raphson method. The method is to start with an initial term x_0 and calculate the iteration sequence given by the recurrence relation

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \quad (n = 0, 1, 2 \ldots).$$

Each term x_{n+1} in this sequence is the x-coordinate of the point where the tangent at $(x_n, f(x_n))$ to the graph of f cuts the x-axis, as illustrated in Figure 5.1. Usually each term of the sequence is a better approximation to a solution of $f(x) = 0$ than the preceding term.

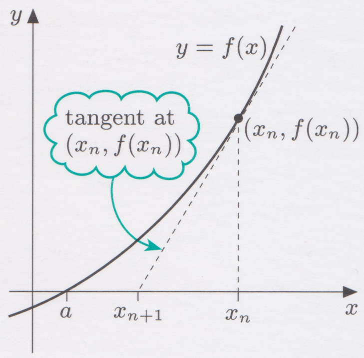

Figure 5.1

The next activity shows how the Newton–Raphson method can be applied in Mathcad to an equation that you saw in the main text.

See Chapter C1, Activities 4.1 and 4.3.

Activity 5.4 The Newton–Raphson method on the computer

Open Mathcad file **221C1-03 The Newton–Raphson method**. The worksheet consists of a single page, in which the terms of a Newton–Raphson iteration sequence are calculated and displayed on a graph and in a table (of the last eleven terms).

The page is set up with the function f defined by $f(x) = x^3 - 2x - 2$. The number of iterations, N, is set to 0, and the initial term, x_0, is set to 2.

The worksheet uses the name Df rather than f' for the derived function of f in the Newton–Raphson recurrence relation. The definition of $Df(x)$ is near the top of the page; it uses the $\frac{d}{dx}$ operator to obtain a formula for the derivative of $f(x)$.

The notation Df is used rather than f' because the prime symbol is difficult to see in Mathcad when placed immediately to the right of an 'f'. (More information about the definition of $Df(x)$ is given in the Mathcad note overleaf.)

(a) (i) Set $N = 1$. The first iteration of the Newton–Raphson method is displayed on the graph. The first two terms x_0 and x_1 of the resulting sequence are displayed in a table, together with the values of f at these points.

(ii) Set $N = 2$ to see the next iteration.

(iii) Set $N = 5$. The resulting further iterations cannot be seen on the graph because the construction lines are very close to the point where the curve crosses the x-axis. However, the values of the terms calculated are displayed in the table.

Use these values to state an approximation for the zero of f which can be seen on the graph.

(b) (i) Change the graph x-axis limit $X1$ from 1.6 to -1.6, to display more of the graph of f. Then set $x_0 = 0.8165$. Describe what you observe.

(ii) Set $N = 30$. The table now shows the terms x_{20} to x_{30}, and you should notice a new construction line within the graph box. Now increase the number of iterations until the iteration sequence seems to settle down to the zero of f.

Find the smallest value of n for which $f(x_n) = 0$ to nine decimal places.

You considered the initial terms $x_0 = 0.8165$ and $x_0 = 0$ in Activity 4.3 of the main text.

(c) Set $x_0 = 0$ and describe what you observe.

Solutions are given on page 29.

Mathcad notes

In the worksheet, the function Df is defined as the result of a symbolic calculation; for example,

$$Df(x) := \frac{d}{dx}f(x) \ \rightarrow \ 3x^2 - 2.$$

This expression may look a little strange, but it is just a combination of some basic Mathcad features and operators. It consists of a standard function definition for $Df(x)$, the $\frac{d}{dx}$ operator to calculate the derivative of f, symbolic evaluation $\rightarrow$ and the symbolic result. It is this result which becomes the right-hand side of the definition for $Df(x)$. If any change is made to f, then Df will change also.

In Activity 5.4(b) and (c), you saw examples of difficulties with the Newton–Raphson method. You should be reassured, however, that such cases are the exception rather than the rule. For most functions and initial terms, the Newton–Raphson method produces a sequence that rapidly approaches a zero of the function. In particular, the method is very reliable if you choose an initial term x_0 that is fairly close to the zero which you seek.

You can choose a suitable value for x_0 by using a sketch or Mathcad plot of the graph of the function.

Activity 5.5 Using the Newton–Raphson method

You should still be working with Mathcad file 221C1-03.

For each of the functions f in parts (a)–(c) below, use the Mathcad worksheet to find approximations for all the solutions of the equation $f(x) = 0$ in the given interval.

In each part, start by resetting the number of iterations N to 0, then edit the definition of the function f to set it to the given function, and set the graph x-axis limits to the endpoints of the given interval. Then choose an appropriate initial term x_0, and set N large enough to display a term x_n for which $f(x_n) = 0$ to nine decimal places.

For some values of x_0 and N you may find that an error occurs in the calculation for the iteration sequence. If this happens, then reset N to 0 (temporarily) to ensure that the graph is drawn, and choose a different value for x_0. See the Mathcad notes for a possible explanation of the problem.

Where there is more than one zero in the given interval, you will need to choose an appropriate initial term x_0 to find each one.

(a) $f(x) = x^2 - x - 1$, in the interval $[1, 2]$

(b) $f(x) = x^3 - x^2 - 5x + 3$, in the interval $[-3, 3]$

(c) $f(x) = x + e^x$, in the interval $[-1, 0]$

Solutions are given on page 29.

Comment

It is possible for a Newton–Raphson iteration sequence with initial term x_0 to converge to a zero other than the zero closest to x_0. For example, in part (b) the initial value $x_0 = 1.5$ lies between the two positive zeros of f but the sequence x_n converges to the negative zero.

Mathcad notes

An error can occur if some term x_n is a stationary point of f. In this case $f'(x_n) = 0$ and so the Newton–Raphson method fails. When this occurs, the expression $f(x_n)/Df(x_n)$ is highlighted in red in the worksheet. Clicking on this expression reveals the error message 'Found a singularity while evaluating this expression. You may be dividing by zero.'.

Now close file 221C1-03.

Chapter C2, Section 5
Integration with the computer

Mathcad can be used to perform integration. In Subsections 5.1 and 5.2 you will use Mathcad to find indefinite and definite integrals, respectively. There is one prepared Mathcad file for these subsections, which provides 'integration templates', but you will also create your own worksheets.

Subsection 5.3 is *optional*; it involves using a prepared Mathcad file to explore estimates for the volumes of solids of revolution obtained by summing the volumes of cylinders.

5.1 Finding indefinite integrals

In Mathcad, indefinite integrals are found using the $\int$ operator. Like the $\frac{d}{dx}$ operator, the $\int$ operator can be used with Mathcad's symbolic commands, to find an algebraic expression for an integral of a given function.

In this subsection you are invited to find indefinite integrals for a variety of functions. Activity 5.1 repeats an activity in the course MST121 in which the Mathcad approach to finding indefinite integrals is introduced. If you have recently studied Chapter C2 of MST121, then you should not find it necessary to work through Activity 5.1 in detail, though you may wish to refresh your memory. This also applies to the comments on the activity.

Activity 5.1 How to find indefinite integrals

In this activity you will use Mathcad to find the indefinite integral of x^2.

The buttons referred to below are on the 'Calculus' and 'Symbolic' toolbars. If you wish to use these and they are not already visible, then either click on the appropriate buttons on the 'Math' toolbar, or select the **View** menu, **Toolbars** and choose **Calculus**, then repeat and choose **Symbolic**.

(a) Create a new (Normal) worksheet.

(b) Enter the $\int$ operator, either by clicking on the $\int$ button on the 'Calculus' toolbar, or by using the keyboard alternative `[Ctrl]i`.

(c) Enter the expression to be integrated, x^2, in the first placeholder after the integral sign. (This expression is called the integrand.) Then enter the variable of integration, x, in the placeholder after the 'd'.

(d) Click on the $\rightarrow$ button ('Symbolic Evaluation') on the 'Symbolic' toolbar, or use `[Ctrl].`, the keyboard alternative. Then click elsewhere on the page, or press `[Enter]`, to obtain the integral. Check that the answer provided by Mathcad is what you expect.

(e) Now go through the same procedure to evaluate the integral $\displaystyle\int u^2\,du$.

(f) If you wish to save your work, then select the **File** menu and use **Save As...** to name and save your worksheet. (It is a good idea to insert a title in your worksheet. If you need to create space for this, then do so by positioning the red cross cursor at the top of the worksheet and pressing `[Enter]` to insert as many blank lines as required.)

If necessary, see Activity 6.3(a) for Chapter A2 in Computer Book A, or consult *A Guide to Mathcad*.

Be careful not to confuse the $\int$ button with the $\int_a^b$ button, which is used for finding definite integrals (as you will see later).

Comment

- ◇ Notice that the $\int$ operator in Mathcad gives only *an* integral (that is, an antiderivative) of the integrand. It does not give *the indefinite* integral because it does not add an arbitrary constant.
- ◇ The outcomes from integrating $\int x^2\,dx$ and $\int u^2\,du$ demonstrate that the form of the indefinite integral depends on the nature of the function being integrated but not on the choice of symbol for the variable of integration.

Mathcad notes

- ◇ In part (e), it is sufficient simply to edit the integral expression $\int x^2\,dx$, replacing each 'x' by a 'u'.
- ◇ You can also find an integral of a function f by first defining f and then evaluating symbolically the expression $\int f(x)\,dx$.
- ◇ The behaviour of the $\int$ operator differs somewhat from that of the $\frac{d}{dx}$ operator, used in Chapter C1. The result of evaluating the $\int$ operator symbolically is *not* affected by a value being defined for the integration variable (x, say) earlier in the worksheet. Furthermore, the $\int$ operator *cannot* be evaluated numerically, since this makes no sense in the context of finding an indefinite integral. If you try this, then Mathcad responds with the word 'function' to the right of the equals sign.

In the next activity, you should continue with the worksheet created in Activity 5.1. (If you did not work through Activity 5.1, then create a new (Normal) worksheet for Activity 5.2.)

Activity 5.2 Finding indefinite integrals

Each of parts (a)–(c) below gives an indefinite integral that you were asked to find in the main text, together with the answer provided in the solutions.

In each part, use Mathcad to find an integral, and compare the answer given by Mathcad with the answer stated here.

If necessary, refer to Activity 5.1(b)–(d).

(a) $\displaystyle\int (u+2e^{7u})\,du = \tfrac{1}{2}u^2 + \tfrac{2}{7}e^{7u} + c$

See Activity 1.3(c).

(b) $\displaystyle\int \cos^2 x\,dx = \tfrac{1}{4}\sin(2x) + \tfrac{1}{2}x + c$

See Activity 1.3(e).

(c) $\displaystyle\int \frac{x^2}{1+x^3}\,dx = \tfrac{1}{3}\ln|1+x^3| + c$

See Activity 3.3(e).

In part (b), $\cos^2 x$ should be input as $\cos(x)^2$; for example, type `cos(x)∧2`.

Solutions are given on page 30.

Comment

- ◇ After the '$+c$' has been added, the Mathcad answers to parts (a) and (b) agree with those obtained in the main text, and that for part (c) is equivalent when $1+x^3>0$. In part (a), recall that $\exp(7u)$ is an alternative way of writing e^{7u}.
- ◇ Recall (from the context of obtaining derivatives symbolically) that the symbolic keywords 'simplify' and 'factor' can be used as alternatives to $\rightarrow$. When applied to integrals, either of these may again provide an outcome in a more convenient form.

See the final Comment item for Chapter C1, Activity 5.1 on page 6 of this computer book.

Save the worksheet that you have created, if you wish. Then close the file.

Mathcad provides a useful means of checking integrations that you have carried out by hand. However, an answer supplied by Mathcad can look quite different from an answer obtained by hand, even if you use Mathcad to simplify the expression. The next activity gives you practice in checking that answers supplied by Mathcad are equivalent to answers obtained by hand.

The Mathcad file associated with this activity provides a 'template for indefinite integrals' that you may also wish to use in other circumstances.

Activity 5.3 Further indefinite integrals

Open Mathcad file **221C2-01 Integration templates**, and turn to page 2 of the worksheet. This page can be used to find an integral (antiderivative) of any function $f(x)$ that you input.

Note that the output from applying the symbolic keywords 'simplify' and 'factor' to the integral of a function is provided, as well as that obtained by straightforward symbolic evaluation.

Each of parts (a)–(c) below gives an indefinite integral that you were asked to find in the main text, together with the answer provided in the solutions. In each case, enter the formula for the integrand into the definition of $f(x)$ in the template, and note the results. If it is not clear that one of the answers supplied by Mathcad is equivalent to the stated expression for $\int f(x)\,dx$ (apart from omission of the '$+c$' in Mathcad), then copy the closest-looking Mathcad answer onto a piece of paper and try to carry out further algebraic manipulation by hand to confirm the equivalence. (However, do not spend long on this for part (c).)

(a) $\displaystyle\int x^3 \ln x\,dx = \tfrac{1}{16}x^4(4\ln x - 1) + c$

See Activity 2.7(a).

(b) $\displaystyle\int \frac{2x+3}{(x+2)^{1/3}}\,dx = \tfrac{6}{5}(x+2)^{5/3} - \tfrac{3}{2}(x+2)^{2/3} + c$

See Activity 3.5(c).

(c) $\displaystyle\int \frac{1}{(1+x^2)^2}\,dx = \tfrac{1}{2}\arctan x + \tfrac{1}{4}\sin(2\arctan x) + c$

See Activity 3.6.

Comment

(a) The Mathcad answer resulting from 'factor' is equivalent to the stated answer.

(b) The Mathcad answer from just symbolic evaluation is equivalent to the stated answer. It is less obvious that this is the case for the Mathcad output from 'simplify' and 'factor', but algebraic manipulation, after taking $(x+2)^{2/3}$ as a common factor in both terms of the first expression, confirms that this is so. The 'factor' output looks the simplest.

(c) Symbolic evaluation in Mathcad provides the answer

$$\int \frac{1}{(1+x^2)^2}\,dx \to \frac{1}{2}\frac{x}{(1+x^2)} + \frac{1}{2}\,\mathrm{atan}(x),$$

and the output from 'simplify' and 'factor' is also close to this. Comparison with the given answer shows that the Mathcad output includes the expression

$$\frac{1}{2}\frac{x}{(1+x^2)} \quad \text{instead of} \quad \tfrac{1}{4}\sin(2\arctan x).$$

Note that Mathcad uses the notation 'atan' for arctan.

It is possible that your output will display terms in a different order to that shown here.

It is not immediately clear that these two expressions are equivalent, but in fact they are, as the following argument shows. For any $x \in \mathbb{R}$, let $\theta = \arctan x$. Then $x = \tan\theta$, so

This argument uses the trigonometric identities

$$1 + \tan^2\theta = \sec^2\theta$$

and

$$\sin(2\theta) = 2\sin\theta\cos\theta.$$

$$\begin{aligned}\frac{1}{2}\left(\frac{x}{1+x^2}\right) &= \frac{1}{2}\left(\frac{\tan\theta}{1+\tan^2\theta}\right)\\ &= \frac{1}{2}\left(\frac{\sin\theta/\cos\theta}{\sec^2\theta}\right)\\ &= \tfrac{1}{2}\sin\theta\cos\theta\\ &= \tfrac{1}{4}\sin(2\theta) = \tfrac{1}{4}\sin(2\arctan x).\end{aligned}$$

This example illustrates that it may be difficult to tell whether an answer supplied by Mathcad is equivalent to one found by hand.

You can obtain evidence that two expressions *may* be equivalent by evaluating them at a few values of x, or by plotting both of them on the same Mathcad graph.

The next activity demonstrates that Mathcad's facility for finding indefinite integrals is not always as helpful as you might hope.

Activity 5.4 Mathcad does not always give the 'best' answer!

You should still be working with Mathcad file 221C2-01, on page 2 of the worksheet.

See Chapter C2, Example 3.2.

(a) In the main text, the indefinite integral

$$\int (3+\sqrt{x})^9\,dx = \tfrac{2}{11}(3+\sqrt{x})^{11} - \tfrac{3}{5}(3+\sqrt{x})^{10} + c$$

was obtained. Find this indefinite integral using Mathcad, and comment on the result.

(b) The result

$$\int \frac{1}{x+\sqrt{x}}\,dx = 2\ln(1+\sqrt{x}) + c \quad (x>0),$$

can be found by hand, using the substitution $u = 1 + \sqrt{x}$. Find this indefinite integral using Mathcad, and comment on the result.

(c) Use Mathcad to find the indefinite integral

$$\int x^n\,dx.$$

Comment on any problems with Mathcad's answer that you notice.

Comment

(a) Mathcad (with any of the three forms of output) gives a lengthy expression for an antiderivative, even though there is a simpler expression found by hand. If these two expressions are both correct, then they must differ by a constant (possibly zero). To check this, you can enter the simpler expression into the Mathcad worksheet, apply the symbolic keyword 'expand' and compare the coefficients of all non-constant terms.

This lengthy expression is obtained by first applying the Binomial Theorem to the integrand and then integrating each term of the resulting expansion.

(b) Once again, Mathcad gives an expression that is more complicated than the one found by hand. In each case, the Mathcad response is

$$\ln(x-1) + 2\,\text{atanh}(x^{1/2}).$$

Now 'atanh' is Mathcad notation for the function $\tanh^{-1}$, the inverse of the hyperbolic tangent function

$$\tanh x = \frac{e^x - e^{-x}}{e^x + e^{-x}}.$$

You will see more about hyperbolic functions in Chapter C3.

It is possible to show that the inverse function of tanh is given by

$$\tanh^{-1}(y) = \tfrac{1}{2}\ln\left(\frac{1+y}{1-y}\right) \quad (-1 < y < 1),$$

so we have

$$2\tanh^{-1}(x^{1/2}) = \ln\left(\frac{1+x^{1/2}}{1-x^{1/2}}\right) \quad (0 \leq x < 1).$$

As well as the relatively complicated nature of the Mathcad response, there is another, less obvious, problem here. Since the domain of the real function ln is $(0, \infty)$, the first term, $\ln(x-1)$, is defined only for $x > 1$. The second term, from the last equation above, has domain $0 \leq x < 1$. Thus it seems that there is no value of x for which both terms are defined!

In fact, the answer given by Mathcad does make sense. It is possible to extend the definitions of the natural logarithm function and the square root function to domains and codomains that include some non-real *complex* numbers. With these extended definitions, the natural logarithm and the square root of each negative real number are defined as complex numbers. So the answer given by Mathcad can be evaluated for most real numbers x; its value differs from the value of $2\ln(1+\sqrt{x})$ by a constant that is a complex number.

You will meet complex numbers in Chapter D1.

However, the answer given by Mathcad is unhelpful, in that a real-valued antiderivative exists (for $x > 0$) but is not found.

(c) Mathcad gives an answer that is correct for all values of n except $n = -1$, for which the answer provided involves a division by zero.

This highlights the fact that, when asked to obtain general results, Mathcad does so without regard to possible exceptions to the rule.

In the function definition $f(x) := x^n$ the variable n will appear in red, as it has not been defined. However, Mathcad ignores this error and is still able to evaluate $\int f(x)\,dx$ symbolically.

Now close file 221C2-01.

5.2 Finding definite integrals

In Mathcad, definite integrals are found using the $\int_a^b$ operator. This operator can be used in either of two ways.

◇ *Symbolically*: Mathcad finds an algebraic expression for an antiderivative of the integrand, evaluates this at the upper and lower limits of integration, and subtracts the second value from the first.

◇ *Numerically*: Mathcad uses a numerical method to find an approximate value for the definite integral.

Examples of both techniques are given in this subsection.

In Activities 5.5–5.7 below, as in the first part of Subsection 5.1, you are asked to work in a new worksheet that you have created yourself.

You will need to decide whether you wish to save this worksheet.

Activity 5.5 introduces you to the symbolic use of the $\int_a^b$ operator. If you have recently studied Block C of MST121, then you should not need to do this activity, as it repeats an activity in that course, though you may wish to refresh your memory.

Activity 5.5 How to evaluate definite integrals

In this activity you will use Mathcad to evaluate the definite integral

$$\int_2^3 \frac{1}{x}\,dx.$$

(a) Enter the $\int_a^b$ operator in your worksheet, either by clicking on the $\int_a^b$ button on the 'Calculus' toolbar, or by using the keyboard alternative `&` (the ampersand sign, given on the keyboard by `[Shift]7`).

(b) Enter the integrand, the variable of integration, and the lower and upper limits of integration in the appropriate placeholders.

The `[Tab]` key provides a good way of moving around the placeholders.

(c) Click on the $\rightarrow$ button ('Symbolic Evaluation') on the 'Symbolic' toolbar, or use `[Ctrl].`, the keyboard alternative. Then click elsewhere on the page, or press `[Enter]`. You should obtain the answer

$$\ln(3) - \ln(2).$$

(d) Click anywhere on this answer, and then enter `=`, either by clicking on the `=` button on the 'Calculator' toolbar or by typing `=`, to evaluate the answer numerically. You should obtain the answer 0.405, which is the value of $\ln 3 - \ln 2$ to three decimal places.

Comment

Evaluating symbolically an expression involving the $\int_a^b$ operator gives an expression which is an *exact* answer (unless the original expression contains a decimal point; see the second Mathcad note below). In the example in this activity the expression is $\ln(3) - \ln(2)$. You can display the decimal value of such an expression by evaluating it numerically. This is done by selecting the expression, and then entering `=`.

Mathcad notes

◇ When you evaluate numerically an expression in Mathcad, the number of decimal places displayed is determined by the value of 'Number of decimal places'. The default value of this is 3, but you can change it by choosing **Result...** from the **Format** menu and then the 'Number Format' tab.

◇ If a Mathcad expression involving the $\int_a^b$ operator has a decimal point in any constant in the integrand, or in *both* limits of integration, then evaluating the expression symbolically gives a decimal answer with up to 20 decimal places. (Such an answer is unaffected by the value of 'Number of decimal places'.) For example, evaluating symbolically the integral below in Mathcad gives the outcome

$$\int_2^3 \frac{1.0}{x}\,dx \rightarrow .40546510810816438198.$$

In the next activity you will use the $\int_a^b$ operator to check the answers to some definite integrals that you were asked to evaluate by hand in the main text.

Activity 5.6 Evaluating definite integrals

Each of parts (a)–(d) below gives a definite integral that you were asked to evaluate in the main text. In each case, use symbolic evaluation in Mathcad to find an exact answer, and then evaluate this answer numerically, to obtain a value to three decimal places.

If necessary, refer to the instructions in Activity 5.5.

(Remember that π can be obtained from a button on the 'Calculator' or 'Greek' toolbar, or by typing `[Ctrl][Shift]p`.)

(a) $\displaystyle\int_0^1 \sin(\pi x)\,dx$ — See Activity 1.5(a).

(b) $\displaystyle\int_{-\pi/4}^{\pi/4} \sec^2 t\,dt$ — See Activity 1.5(c).

(c) $\displaystyle\int_1^2 xe^{2x}\,dx$ — See Activity 2.4(c).

(d) $\displaystyle\int_{\pi/12}^{\pi/6} x\sin(3x)\,dx$ — See Activity 2.6(b).

Solutions are given on page 30.

In the next activity you will see an example of a definite integral that Mathcad is unable to evaluate symbolically. When this happens it is often worth attempting to evaluate the integral numerically, and the activity shows you how to do this. If you have recently studied Block C of MST121, then you should not need to do Activity 5.7, as it repeats an activity in that course, though you may wish to refresh your memory.

Activity 5.7 An awkward definite integral

(a) Use Mathcad to try to evaluate symbolically the definite integral

$$\int_0^1 e^{-t^3}\,dt.$$

You should find that the definite integral is repeated without alteration (except for the replacement of e^{-t^3} by $\exp(-t^3)$). This means that Mathcad has been unable to calculate an algebraic expression for an antiderivative of the integrand, and so it cannot evaluate symbolically the given definite integral.

(b) Evaluate the definite integral numerically, as follows. Click anywhere on the expression just created, and then enter `=`.

You should find that the answer 0.808 is displayed.

Comment

◇ To evaluate a definite integral by direct numerical evaluation, you can enter it in the same way as for symbolic evaluation, then enter `=`.

◇ The reason why some definite integrals can be evaluated numerically but not symbolically in Mathcad is that symbolic evaluation requires Mathcad to find an algebraic expression for an antiderivative, whereas numerical evaluation involves the use of a numerical algorithm. The answer obtained from this algorithm is an approximation, though usually an accurate one.

Mathcad notes

On rare occasions, the numerical method used by Mathcad for evaluating definite integrals fails to produce a value. In such a case, the integral is marked with the error message 'Can't converge to a solution.'.

Save the worksheet that you have created, if you wish. Then close the file.

The next activity requires the 'integration templates' Mathcad file that you used in the latter part of Subsection 5.1, but this time for application to definite integrals.

Activity 5.8 Further definite integrals

Open Mathcad file **221C2-01 Integration templates**, and turn to page 3 of the worksheet. This page can be used to evaluate the definite integral of $f(x)$ from a to b, for any function $f(x)$ and limits of integration a, b that you input.

Note that three outputs are provided: for symbolic evaluation alone, for symbolic evaluation whose exact result is then evaluated as a decimal, and for direct numerical evaluation.

You may prefer to work in manual calculation mode here, to define f, a, b and then calculate the results. (See the first Mathcad note at the top of page 7.)

Evaluate each of the following integrals. (Enter the formula for each integrand into the definition of $f(x)$ in the template, enter the limits of integration into the definitions of a and b, and note the results.)

(a) $\displaystyle\int_0^1 \cos(\sin x)\,dx$ (b) $\displaystyle\int_0^{\pi} \cos(x^2)\,dx$ (c) $\displaystyle\int_0^{\pi} \cos^2 x\,dx$

Solutions are given on page 30.

Comment

- Symbolic evaluation does not lead to a decimal answer in either part (a) or (b), though direct numerical evaluation works in both cases. In part (a), Mathcad does not find an antiderivative of the integrand, and indicates this by repetition of the expression for the integral entered.

  In part (b), an antiderivative *is* found, but it involves the obscure function 'FresnelC'. Attempting to evaluate this numerically causes the expression to be highlighted in red, and clicking on this expression reveals the error message 'This variable or function is not defined above.'. The function FresnelC is one of a small number of functions that are recognised by Maple (Mathcad's symbolic processor), but which cannot be evaluated by Mathcad.
- The integral in part (c) can be evaluated either symbolically or by direct numerical evaluation. Symbolic evaluation gives $\frac{1}{2}\pi$.

In Mathcad many definite integrals can be evaluated either symbolically or by direct numerical evaluation. You might wonder which it is more appropriate to invoke in any given situation. If you want an exact answer (so that you can see where constants such as π feature in it, for example), or if a general result is required, such as a formula for

$$\int_0^1 \cos(kx)\,dx \quad \text{(where } k \text{ is a non-zero constant),}$$

then use the symbolic approach. If you simply want a decimal number that is an accurate value for the definite integral, then numerical evaluation should suffice. The definite integral template in Mathcad file 221C2-01 provides for all eventualities, giving where available both an exact value and a decimal value for the answer. When direct numerical evaluation and symbolic evaluation lead to the same decimal output, this is good evidence that the answers provided are correct.

For example, the values of all the definite integrals in Activity 5.6 can be found satisfactorily using direct numerical evaluation.

This template can also be used to obtain symbolic expressions for definite integrals in which the interval end-points are not given as specific numbers, provided that the definitions for a and b are either deleted or temporarily moved below the output results. For example, under these circumstances, the function $f(x) = \cos^2 x$ input for Activity 5.8(c) gives the output

$$\int_a^b f(x)\,dx \;\to\; \tfrac{1}{2}\cos(b)\sin(b) + \tfrac{1}{2}b - \tfrac{1}{2}\cos(a)\sin(a) - \tfrac{1}{2}a.$$

However, neither symbolic nor direct numerical evaluation can be guaranteed always to give an accurate answer, as the following activity shows.

Activity 5.9 Mathcad may not give the right answer!

In each case below, try to evaluate the definite integral given by hand. Then use Mathcad to try and evaluate the given integral.

You should still be working with Mathcad file 221C2-01, on page 3 of the worksheet.

(a) $\displaystyle\int_{-1}^{1} (x^4)^{1/4}\,dx$

(b) $\displaystyle\int_0^{\pi/4} \sec^2 x\sqrt{\tan x}\,dx$

(c) $\displaystyle\int_{-1}^{1} \frac{1}{x^2}\,dx$

(d) $\displaystyle\int_{10^{-5}}^{1} \frac{1}{x^2}\cos\left(\frac{1}{x}\right)\,dx$

Comment

(a) The integrand $(x^4)^{1/4}$ is an alternative way of writing $|x|$, so the definite integral is

$$\begin{aligned}\int_{-1}^{1} |x|\,dx &= \int_{-1}^{0} (-x)\,dx + \int_0^1 x\,dx\\ &= [-\tfrac{1}{2}x^2]_{-1}^{0} + [\tfrac{1}{2}x^2]_0^1 = 1.\end{aligned}$$

Direct numerical evaluation in Mathcad gives this answer, while symbolic evaluation gives the (wrong) answer 0.

The cause of the error in the symbolic evaluation is that Mathcad finds an antiderivative of the integrand which is incorrect for $x < 0$.

(b) Using the substitution $u = \tan x$, we find that

$$\begin{aligned}\int_0^{\pi/4} \sec^2 x\sqrt{\tan x}\,dx &= \int_0^1 u^{1/2}\,du\\ &= [\tfrac{2}{3}u^{3/2}]_0^1 = \tfrac{2}{3}.\end{aligned}$$

Mathcad fails to find an antiderivative in this case, though direct numerical evaluation gives the answer correct to three decimal places.

(c) The following 'calculation' is *not* correct!

$$\int_{-1}^{1} \frac{1}{x^2}\, dx = \left[-\frac{1}{x}\right]_{-1}^{1} = -1 - (-(-1)) = -2.$$

The integrand $1/x^2$ is not defined for $x = 0$, and so we should not attempt to integrate it over any interval, such as $[-1, 1]$, which contains 0.

However, Mathcad does supply answers! Symbolic evaluation gives the answer ∞, which provides some indication that there is a difficulty in evaluating the integral. The wrong answer from direct numerical evaluation, 1.376×10^3, provides no such warning.

(d) Using the substitution $u = 1/x$, we find that

$$\begin{aligned}\int_{10^{-5}}^{1} \frac{1}{x^2} \cos\left(\frac{1}{x}\right) dx &= -\int_{10^5}^{1} \cos u\, du \\ &= \int_{1}^{10^5} \cos u\, du \\ &= [\sin u]_1^{10^5} = \sin(10^5) - \sin 1 \simeq -0.806.\end{aligned}$$

Symbolic evaluation in Mathcad agrees with this answer, while direct numerical evaluation fails to produce a result. (The problem here is that the integrand given has a large number of oscillations packed into the left-hand end of the interval of integration.)

Mathcad notes

The infinity symbol ∞ can be obtained from a button on the 'Calculus' toolbar, or by typing `[Ctrl][Shift]z`. If you evaluate it numerically, then Mathcad gives the value 1×10^{307}, which is the largest power of 10 that Mathcad can handle.

Now close file 221C2-01.

Usually, Mathcad gives accurate and helpful responses when asked to integrate. However, you have just seen some examples which show that Mathcad's results sometimes need to be treated with caution. Just occasionally, its numerical calculations may give an inaccurate answer. Sometimes, its symbolic responses are not totally correct or are in a form that is not helpful.

Similar remarks apply to *any* numerical or symbolic manipulation package.

The broad message to take away from this experience is that any results obtained from Mathcad should be treated critically. For example, when you evaluate an integral numerically, you should try to estimate the answer by alternative means, so that you have something against which to compare the Mathcad result.

You also saw, in Activity 5.9(c), that hand calculation sometimes needs to be approached with care. Watch out for points in the proposed interval of integration at which the integrand is undefined, and do not integrate a function over any interval in which the function is not continuous.

5.3 Volumes of solids of revolution (Optional)

In this subsection you can use a prepared Mathcad file to explore the volumes of solids of revolution. The solid of revolution generated by the region bounded by the graph of a continuous function f from a to b is illustrated in Figure 5.1.

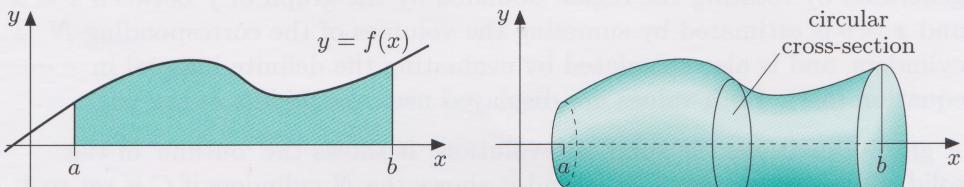

Figure 5.1 Volume of revolution

In the main text you saw a derivation of the following result.

See Chapter C2, Section 4.

> **Volume of solids of revolution**
>
> If f is any function that is continuous on $[a, b]$, then the volume of the solid of revolution generated by the region bounded by the graph of f from $x = a$ to $x = b$ is given by
>
> $$\text{volume of revolution} = \pi \int_a^b (f(x))^2\, dx. \qquad (5.1)$$

You also saw that the volume of a solid of revolution can be approximated by summing the volumes of a set of cylinders. Figure 5.2 shows a set of N cylinders whose total volume approximates the volume of the solid in Figure 5.1.

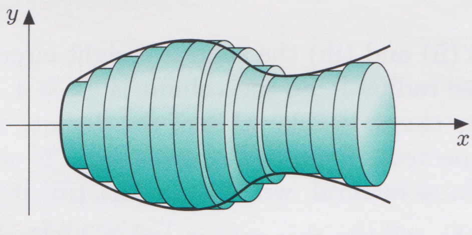

Figure 5.2 'Sum of cylinders'

Each cylinder has 'width' h, where $h = (b - a)/N$, and thus has its left-hand edge at the point $x = a + ih$, where the value of i is 0 for the left-most cylinder, 1 for the next cylinder, and so on, up to $N - 1$ for the final cylinder. For $i = 0, 1, \ldots, N - 1$, the radius of the ith cylinder is $f(a + ih)$. The total volume of the cylinders is therefore given by

$$\pi h \sum_{i=0}^{N-1} (f(a + ih))^2, \quad \text{where } h = (b - a)/N. \qquad (5.2)$$

In general, the greater the number N of cylinders, the more closely sum (5.2) approximates the volume of the solid.

In the following, *optional*, activity you can use a prepared Mathcad file to explore such approximations for volumes of solids of revolution.

Activity 5.10 Volumes of solids of revolution (Optional)

Open Mathcad file **221C2-02 Volumes of solids of revolution**. The worksheet consists of a single page.

A function f, constants a and b, and a number N of subintervals are defined near the top of the page. The volume of the solid of revolution generated by rotating the region bounded by the graph of f between $x = a$ and $x = b$ is estimated by summing the volumes of the corresponding N cylinders, and is also calculated by evaluating the definite integral in equation (5.1). Both values are displayed near the middle of the page.

A graph illustrates the solid of revolution. It shows the 'outline' of the solid when the variable C is 0, and it shows the N cylinders if C is set to 1.

The cylinders are not drawn if N is greater than 20 because there is a limit to the number of points that Mathcad can plot on a graph.

(a) Initially $f(x) = \frac{1}{4}x$, $a = 0$ and $b = 4$, so the solid is a right circular cone with height 4 and base radius 1. The number of cylinders is $N = 4$.

(i) Set $C = 1$ to see the cylinders. You will find that only three of the four cylinders are visible because the radius of the first is zero.

(ii) Set N to 5, 10, 100 and 1000 in turn, and observe the effect on the estimate obtained by summing the volumes of the cylinders.

If you have a fast computer, then you may also like to try $N = 20$, to see the maximum number of cylinders that the worksheet can display.

(iii) Set $f(x) = 1 - \frac{1}{4}x$ and repeat part (a)(ii).

(b) Set $f(x) = \sin x$ and $b = \pi$; keep $a = 0$. Try varying the value of N, and observe the effect on the estimate.

(c) You may like to use the worksheet to explore, and to check the answers for, some of the other examples given in Section 4 of the main text. For example, if you would like to explore the volume of a sphere of radius 1, then set $f(x) = \sqrt{1 - x^2}$, $a = -1$ and $b = 1$. You will have to resize the graph if you want the sphere to look spherical.

The volume of a sphere of radius r was determined in Activity 4.1 in the main text.

You may also like to try the function $f(x) = 1 + \frac{1}{2}\sin x$ over the interval $[0, 6]$, which gives a pleasing vase shape! You can alter the shape of the vase by varying the definition of f.

Comment

(a) In each of parts (ii) and (iii) the solid is a right circular cone with height 4 and base radius 1. So the volume is $\frac{4}{3}\pi \cong 4.189$. In both cases, the estimate gets closer to this value of the definite integral as N is increased, as expected. In part (ii) the estimate is always less than the value of the definite integral, whereas in part (iii) it is always greater.

This volume was found, using the formula, in Example 4.1 in the main text.

(b) The volume of the solid in this case is $\frac{1}{2}\pi^2 \cong 4.935$. You may have been surprised to find that the estimate always appears to be *equal* to the value of the definite integral in this case (except when $N = 1$). This is actually true, and the proof is not hard, but it would take too much space to give it here.

This volume was found in Activity 4.2(b) in the main text.

Mathcad notes

◇ The summation sign is obtained from the $\sum_{n=1}^{m}$ button on the 'Calculus' toolbar, or by typing `[Ctrl]$` (for which you have to press the three keys `[Ctrl]`, `[Shift]` and 4 together).

◇ The cylinders are filled in by plotting zig-zag lines very close together. Tiny gaps between the lines may appear if the graph is printed.

Now close file 221C2-02.

Chapter C3, Section 5
Taylor series with the computer

Mathcad can be used both to find a given number of terms of a Taylor series and to show how the graphs of the resulting Taylor polynomials compare with that of the original function. In Subsection 5.1 you will see how to find Taylor polynomials for a given function about a specified point, while the graphical approach in Subsection 5.2 illustrates how these polynomials approximate the given function.

The single Mathcad file for this section includes both symbolic and graphical templates for the investigation of Taylor polynomials.

5.1 Using Mathcad to find Taylor series

In this subsection you will learn how to use Mathcad to find the first few terms of a Taylor series.

Activity 5.1 Taylor series in Mathcad

Open Mathcad file **221C3-01 Taylor series and polynomials**. Page 1 introduces the worksheet. Work through page 2, and then turn to page 3.

Page 2 describes how to enter the Mathcad instructions for finding a Taylor polynomial. Rather than expecting you to do this from scratch on each occasion, page 3 of the worksheet provides a 'Taylor series template' which reduces the amount of work involved. This page can be used to find the Taylor polynomial for any function $f(x)$ that you input, provided that the centre a and degree n of the polynomial are also specified. The template is set up with $f(x) = e^x$, $a = 0$ and $n = 5$, giving the same Taylor polynomial that you found on page 2 of the worksheet.

The first four terms agree with the quartic Taylor polynomial for $f(x) = e^x$, found in Example 2.1. The polynomial here also agrees with the first five terms of the Taylor series found in Example 3.1.

Each of parts (a)–(c) below gives a standard function, together with its Taylor series about 0. In each case, enter the formula for the function into the definition of $f(x)$ in the template, choose $n = 8$, and confirm that the polynomial output from Mathcad agrees with the corresponding Taylor series for each term shown on screen.

These series are stated in Subsection 3.2, in the table on page 32 of the main text.

(a) $\ln(1+x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \frac{1}{5}x^5 - \cdots$

(b) $\sin x = x - \dfrac{1}{3!}x^3 + \dfrac{1}{5!}x^5 - \dfrac{1}{7!}x^7 + \dfrac{1}{9!}x^9 - \cdots$

(c) $\cos x = 1 - \dfrac{1}{2!}x^2 + \dfrac{1}{4!}x^4 - \dfrac{1}{6!}x^6 + \dfrac{1}{8!}x^8 - \cdots$

Comment

◇ The Mathcad output agrees in each case with the given series. For parts (b) and (c), this follows because

$$3! = 6, \quad 4! = 24, \quad 5! = 120, \quad 6! = 720, \quad 7! = 5040, \quad 8! = 40\,320.$$

◇ As noted in the main text, $\sin x$ is an odd function, and hence its Taylor series has no term involving an even power of x. (In particular, there is no x^8 term, even though the specified polynomial degree here is 8.) Similarly, $\cos x$ is an even function, and so its Taylor series has no term involving an odd power of x.

Mathcad notes

For some Taylor polynomials, Mathcad cannot display this many terms.

Mathcad's symbolic keyword 'series' can be used to find Taylor polynomials up to the term in x^{99}; that is, the largest value you can enter in the second placeholder for 'series' is 100.

In the next activity, you are invited to use Mathcad to find Taylor polynomials in further instances where corresponding results were obtained by hand in the main text.

Activity 5.2 Further Taylor polynomials

You should still be working with Mathcad file 221C3-01, on page 3 of the worksheet.

For each of parts (a)–(d) below, use the worksheet to find the Taylor polynomial specified, and compare your answer with that obtained earlier in the main text and stated again here. (You will need to specify a and n in the worksheet, as well as $f(x)$.)

See Example 2.2.

(a) $f(x) = \ln x$, about $x = 1$, up to the term in $(x-1)^4$

$$p(x) = (x-1) - \tfrac{1}{2}(x-1)^2 + \tfrac{1}{3}(x-1)^3 - \tfrac{1}{4}(x-1)^4$$

See Activity 2.3.

(b) $f(x) = \sin x$, about $x = \frac{1}{6}\pi$, up to the term in $(x - \frac{1}{6}\pi)^3$

$$p(x) = \tfrac{1}{2} + \tfrac{1}{2}\sqrt{3}(x - \tfrac{1}{6}\pi) - \tfrac{1}{4}(x - \tfrac{1}{6}\pi)^2 - \tfrac{1}{12}\sqrt{3}(x - \tfrac{1}{6}\pi)^3$$

See Example 4.4.

(c) $f(x) = e^x \cos x$, about $x = 0$, up to the term in x^3

$$p(x) = 1 + x - \tfrac{1}{3}x^3$$

See Activity 4.10(c). Recall that arcsin should be entered as 'asin' in Mathcad.

(d) $f(x) = \arcsin x$, about $x = 0$, up to the term in x^5

$$p(x) = x + \tfrac{1}{6}x^3 + \tfrac{3}{40}x^5$$

Comment

The Mathcad answer agrees in each case with that in the main text. In part (b), the occurrences of $\sqrt{3}$ in the text answer are replaced by $3^{1/2}$ in the Mathcad output.

You have seen that many Taylor series can be expressed concisely using sigma notation. For example, the Taylor series about 0 for the exponential function is given by

$$e^x = 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \frac{1}{4!}x^4 + \cdots, \quad \text{for } x \in \mathbb{R},$$

which can be written concisely as

$$e^x = \sum_{k=0}^{\infty} \frac{1}{k!}x^k, \quad \text{for } x \in \mathbb{R}.$$

Similarly, the Taylor series about 0 in Activity 5.1 can be expressed as

$$\ln(1+x) = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k}x^k, \quad \text{for } -1 < x < 1;$$

These expressions for $\sin x$ and $\cos x$ were given in Subsection 3.1, on page 32 of the main text.

$$\sin x = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!}x^{2k+1}, \quad \text{for } x \in \mathbb{R};$$

$$\cos x = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!}x^{2k}, \quad \text{for } x \in \mathbb{R}.$$

In the next activity you can use Mathcad to find the first few terms of two Taylor series. You are asked to look for patterns in these terms, to help you in conjecturing expressions in sigma notation for the series.

Activity 5.3 Taylor series in sigma notation

For each of parts (a) and (b) below, use the worksheet to find the Taylor series about 0 for the given function, up to the term in x^5. Then conjecture the next three terms of the series, and use Mathcad to verify your answer. Finally, try to conjecture the whole series, and write down your answer using sigma notation.

You should still be working with Mathcad file 221C3-01, on page 3 of the worksheet.

(a) $f(x) = \dfrac{2-x}{(1-x)^2}$ (b) $f(x) = \dfrac{x^2-x}{(1+x)^3}$

Solutions are given on page 30.

Comment

◇ You could have calculated the first few terms of these Taylor series by hand, though this would have been much slower than using the computer. For example, for part (a) you can use the binomial series to find the Taylor series about 0 for $1/(1+x)^2$, and then substitute $-x$ for x to find the Taylor series about 0 for $1/(1-x)^2$. Multiplying this series by the polynomial $2-x$ then gives the required Taylor series.

The Taylor series for $1/(1+x)^2$ was found in Activity 4.4(a) in the main text.

◇ You can use Mathcad to check your final answers, if you wish. First create some extra space below the final line on page 3 of the worksheet, by placing the red cross cursor there and pressing `[Enter]` to insert some blank lines. Then enter the summation sign by clicking on the $\sum_{n=1}^{m}$ button on the 'Calculus' toolbar, or by typing `[Ctrl]$` (for which you have to press the three keys `[Ctrl]`, `[Shift]` and 4 together). Enter your expression for the kth term of the Taylor series. On the summation sign enter k and the lower limit into the bottom two placeholders, and the symbol ∞ into the top placeholder (either from a button on the 'Calculus' toolbar or by typing `[Ctrl][Shift]z`).

The `[Tab]` key provides a quick way of moving round the placeholders on the summation sign.

To evaluate this expression symbolically, click on the $\rightarrow$ button on the 'Symbolic' toolbar, or use `[Ctrl].`, the keyboard alternative. Then click elsewhere on the page, or press `[Enter]`. If your expression in sigma notation is correct, then Mathcad should respond with an answer that is equivalent to the original expression given in the activity. (If this check does not work, then try replacing ∞ by n in the top placeholder, to check the first few terms of the series.)

Note that Mathcad uses the standard exclamation mark notation for factorials. This is available from the $n!$ button on the 'Calculator' toolbar or by typing `!` (given by `[Shift]1`).

5.2 Graphs of Taylor polynomials

In this subsection, you will see how the graphs of Taylor polynomials approximate that of the original function more and more closely as the degree of the polynomial is increased. By observing this behaviour, it is possible to estimate the 'maximum' ranges of validity of the corresponding Taylor series.

Activity 5.4 Exploring graphs of Taylor polynomials

You should still be working with Mathcad file 221C3-01.

Turn to page 4 of the Mathcad worksheet. This page is designed for plotting the graphs of a function $f(x)$ (in black) and its Taylor polynomial of degree n about a (in red).

(a) The page is set up initially to plot the function $f(x) = e^x$ and its Taylor polynomial of degree 0 about 0.

(i) Increase the value of n to 1, 2, 3 and 4 in turn, and observe the effect on the graph of the Taylor polynomial.

You can see the graphs of all these Taylor polynomials at the same time if you set $m = 5$. Try this now, and reset $m = 1$ afterwards.

You can set m to any integer value between 1 and 5, to display graphs of the m Taylor polynomials of degrees $n - m + 1$ up to n.

(ii) Set the x- and y-axis limits as follows: $X1 = -10$, $X2 = 5$, $Y1 = -5$ and $Y2 = 40$.

(iii) Notice from the graph that the current Taylor polynomial (of degree 4) appears to approximate f closely over an interval which is approximately $[-1.5, 1.5]$.

This interval depends to some extent on the vertical scale of the graph.

Increase n to 10, 20 and 30 in turn, and observe the effect on the interval over which the polynomial appears to approximate f closely.

(b) Now explore the Taylor polynomials about 0 of the function $f(x) = 1/(1 - x)$, by following the instructions below.

(i) Reset $n = 0$, then enter the expression $1/(1 - x)$ into the definition of $f(x)$.

(ii) Set the x- and y-axis limits as follows: $X1 = -5$, $X2 = 5$, $Y1 = -5$ and $Y2 = 5$.

(iii) Increase the value of n to 1, 2, 3, 4, 10, 20 and 30 in turn, and observe the effect on the interval over which the polynomial appears to approximate f closely.

Comment

◇ In part (a), as n increases, the left-hand endpoint of the interval over which the Taylor polynomial appears to approximate the function closely seems to move without limit in the negative direction. It is more difficult to see what happens to the right-hand endpoint, as the graph of the function is steep to the right of the y-axis, but in fact it moves without limit also in the positive direction.

In part (b), as n increases, the left-hand and right-hand endpoints seem to tend to -1 and 1, respectively.

These observations are explained by the fact that the Taylor series about 0 for e^x is valid for all $x \in \mathbb{R}$, whereas the Taylor series about 0 for $1/(1 - x)$ is valid only for x in the interval $(-1, 1)$.

◇ The current Mathcad page includes a feature that can be used for checking calculations in which Taylor polynomials are used to find approximate values of functions at particular points. You carried out calculations of this type in Subsection 2.3 of the main text.

If you set $n = 5$ and $m = 5$ and scroll down to the bottom of the page, then you will see that the Taylor polynomials of degrees 1, 2, 3, 4 and 5 for f about a are evaluated at a particular value of x, namely, $x = 0.25$. As expected, as n increases, these values become progressively closer to the value of $f(x)$, which is also displayed. The corresponding remainders are also shown in a table. You can change the value of x here to find approximations for f at a different point,

and you can increase the value of n to calculate approximations using higher-degree Taylor polynomials. The table contains the values given by the m Taylor polynomials of degrees $n-m+1$ up to n.

Mathcad notes

If you try to plot the graph of a function f using a range variable x, and the formula for the rule of f is not defined at one or more of the values of x, then Mathcad usually avoids the problem by omitting any such value from the range. For example, in part (b) Mathcad is able to plot the graph of the function $f(x)=1/(1-x)$, although the formula gives no value for $x=1$.

However, sometimes rounding errors in Mathcad's internal calculations may alter slightly the values taken by the range variable and cause a spurious vertical line to appear at the 'problem value'. This happens because Mathcad graphs are produced by plotting a point for each value taken by the range variable and then joining successive points with line segments. If the values taken by the function are large and positive for x-values on one side of the problem value, and are large and negative on the other side, then the graph will include an apparently vertical line joining the two 'branches' of the graph. (In this worksheet the step size used for the graph range is set to a power of 2; for example, a step size of $\frac{1}{128}$ $(=2^{-7})$ is used rather than $\frac{1}{100}$. This reduces the likelihood of rounding errors, because such numbers can be stored and manipulated very accurately in binary form inside the computer.)

The next activity is similar to Activity 5.4, but it involves even and odd functions.

Activity 5.5 Taylor polynomials of even and odd functions

In this activity you will explore the graphs of Taylor polynomials of even and odd functions.

You should still be working with Mathcad file 221C3-01, on page 4 of the worksheet.

For each of parts (a)–(c) below, explore the Taylor polynomials about 0 of the function $f(x)$ given, as follows.

(i) Reset $n=0$, then enter the expression given into the definition of $f(x)$.

(ii) Set the x-axis limits $X1$, $X2$ and the y-axis limits $Y1$, $Y2$ to the values indicated.

(iii) Increase the value of n to 1, 2, 3, 4, 10, 20 and 30 in turn, and observe the effect on the interval over which the polynomial appears to approximate f closely.

(a) $f(x)=\cos x;\quad X1=-10,\ X2=10,\ Y1=-1.5,\ Y2=1.5$

(b) $f(x)=\sin x;\quad X1=-10,\ X2=10,\ Y1=-1.5,\ Y2=1.5$

(c) $f(x)=\dfrac{1}{1+x^2};\quad X1=-2,\ X2=2,\ Y1=-0.5,\ Y2=1.5$

You found the Taylor series about 0 for this function in Example 4.1.

Comment

(a) The 'function(f) =' statement in the worksheet now shows "is EVEN" on the right-hand side, whereas in Activity 5.4 the right-hand side was "is neither even nor odd". The function is even because $f(x) = f(-x)$, and the graphs of both the function and its Taylor polynomials are symmetric about the y-axis (unchanged under reflection in the y-axis).

As n increases, the left-hand and right-hand endpoints of the interval over which the Taylor polynomial appears to approximate the cosine function closely seem to move without limit in the negative and positive directions, respectively.

The graphs of the Taylor polynomials of degrees 1 and 3 about 0 are the same as those of degrees 0 and 2, respectively; this is as expected, since the cosine function is even.

(b) The 'function(f) =' statement in the worksheet now shows "is ODD" on the right-hand side. The function is odd because $f(x) = -f(-x)$, and the graphs of both the function and its Taylor polynomials are unchanged by rotation through the angle π about the origin.

As n increases, the left-hand and right-hand endpoints of the interval over which the Taylor polynomial appears to approximate the sine function closely seem to move without limit in the negative and positive directions, respectively.

The graphs of the Taylor polynomials of degrees 2 and 4 about 0 are the same as those of degrees 1 and 3, respectively, as expected, since the sine function is odd.

(c) The function $f(x) = 1/(1+x^2)$ is even. The left-hand and right-hand endpoints of the interval over which the Taylor polynomial appears to approximate this function closely tend to -1 and 1, respectively.

The observations about the endpoints of the intervals in parts (a), (b) and (c) are explained by the fact that the Taylor series about 0 for $\cos x$ and $\sin x$ are both valid for all $x \in \mathbb{R}$, whereas the Taylor series about 0 for $1/(1+x^2)$ is valid only for values of x in the interval $(-1, 1)$.

If the function is even or odd, then the effect of setting m to a value other than 1 differs from that in Activity 5.4. The graphs now plotted are those for the m Taylor polynomials of degrees $n - 2m + 2$ up to n, at intervals of degree 2. However, m graphs are displayed in either case.

In the final activity of this section you will explore two Taylor series that you have not met before in the course. You are asked to use the Mathcad worksheet to estimate their 'maximum' ranges of validity.

Activity 5.6 Exploring ranges of validity

You should still be working with Mathcad file 221C3-01, on page 4 of the worksheet.

For each of the functions in parts (a) and (b) below, decide whether the function is even, odd, or neither. By choosing appropriate values for the x- and y-axis limits, and then increasing the value of n from 0, estimate the 'maximum' range of validity of the Taylor series about 0 for the function.

(a) $f(x) = \dfrac{1}{3-x}$ (b) $f(x) = \dfrac{x}{1+4x^2}$

Solutions are given on page 30.

Now close file 221C3-01.

Solutions to Activities

Chapter C1

Solution 5.1

Where more than one expression is given for a solution here, the first is similar to the Mathcad output, and the second is a form that you are more likely to obtain by hand.

(a) $3x^2 - 12x - 15$

(b) $4\pi r^2$

(c) $\dfrac{(\cos(t) - 2t)}{\exp(t)} - \dfrac{(\sin(t) - t^2)}{\exp(t)}$
$= \dfrac{\cos t - \sin t + t^2 - 2t}{e^t}$

(d) $2\cos(x^2)x = 2x\cos(x^2)$

(e) $-4\sin(4x)$

(f) $2t\ln(t) + \dfrac{(t^2+3)}{t}$

(g) $\dfrac{1}{u(u^2+3)} - 2\dfrac{\ln(u)}{(u^2+3)^2}u = \dfrac{u^2+3-2u^2\ln u}{u(u^2+3)^2}$

(h) $\dfrac{(\exp(t)+1)}{(\exp(t)+t)} = \dfrac{e^t+1}{e^t+t}$

Solution 5.3

(a) Mathcad ('factor') supplies the answer $\exp(x^2)(1+2x^2)$, which is equivalent to the stated answer.

(b) Mathcad ('simplify') supplies the answer

$$\frac{1}{2}\frac{(\cos(x^{1/2})x^{1/2} - \sin(x^{1/2}))}{x^{3/2}},$$

which (since $\sqrt{x} = x^{1/2}$) is equivalent to the stated answer.

(c) Mathcad ($\rightarrow$) supplies an answer equivalent to

$$-\sin[(x+4)\sec x][\sec x + (x+4)\sec x \tan x],$$

which is in turn equivalent to the stated answer, since

$$\sec x + (x+4)\sec x\tan x = \sec x(1 + (x+4)\tan x).$$

Alternatively, use of 'factor' supplies the answer

$$-\sin[(x+4)\sec x]\sec x(1 + (\tan x)x + 4\tan x),$$

which is equivalent to the stated answer, since

$$x\tan x + 4\tan x = (x+4)\tan x.$$

(d) Mathcad ('factor') supplies an answer equivalent to

$$x^3e^x(4\sin x + (\sin x)x + x\cos x).$$

Since we have

$$4\sin x + x\sin x = (4+x)\sin x,$$

this is in turn equivalent to the stated answer.

(e) Mathcad ('simplify' or 'factor') supplies an answer equivalent to

$$\frac{x^3 + x + x^3\tan^2 x + x\tan^2 x - x^2\tan x + \tan x}{(1+x^2)^2}.$$

The numerator here is

$$\begin{aligned} &x^3 + x + x^3\tan^2 x + x\tan^2 x - x^2\tan x + \tan x \\ &= (x^3+x)(1+\tan^2 x) + (1-x^2)\tan x \\ &= x(1+x^2)\sec^2 x + (1-x^2)\tan x \end{aligned}$$

(since $1 + \tan^2 x = \sec^2 x$), showing that the Mathcad output is equivalent to the stated answer.

Solution 5.4

(a) In part (iii), the table confirms that $f(x_4) = 0$ to nine decimal places, so $x_4 = 1.769\,292\,354$ is a good approximation for the zero of f.

(The fact that $f(x_4) = 0$ to nine decimal places does not, however, guarantee that x_4 is an approximation for the zero accurate to nine decimal places, although this is often the case.)

(b) (i) With $x_0 = 0.8165$, the tangent in the first iteration is nearly horizontal because 0.8165 is close to a stationary point of f. This causes the tangent to cross the x-axis at a point far away from the zero that we seek, and so the next term x_1 is very large. In fact, from the table, $x_1 = 184\,399$ to the nearest integer.

(ii) The smallest value of n with $f(x_n) = 0$ to nine decimal places is $n = 34$.

(c) With $x_0 = 0$, the terms of the sequence alternate between 0 and -1; that is, these two values form a 2-cycle. So with this starting value, the sequence never approaches the zero of f.

Solution 5.5

(a) There is one solution in the interval $[1, 2]$ (namely $\frac{1}{2}(1+\sqrt{5})$), whose value is given as 1.618 033 989. (The other solution of the quadratic equation lies outside the given interval.)

(b) There are three solutions in the interval $[-3, 3]$, whose values are given as $-2.086\,130\,198$, $0.571\,993\,268$ and $2.514\,136\,929$.

(c) There is one solution in the interval $[-1, 0]$, whose value is given as $-0.567\,143\,290$.

Chapter C2

Solution 5.2

One difference in each case is that the Mathcad answer does not include an arbitrary constant.

(a) The answer supplied by Mathcad is

$$\tfrac{1}{2}u^2 + \tfrac{2}{7}\exp(u),$$

which is equivalent to the given answer.

(b) The Mathcad answer is

$$\tfrac{1}{2}\cos(x)\sin(x) + \tfrac{1}{2}x.$$

The trigonometric identity $\sin(2x) = 2\sin x \cos x$ shows that the two answers are equivalent.

(c) The Mathcad answer is

$$\tfrac{1}{3}\ln(1 + x^3).$$

The two answers differ only in that the Mathcad answer has brackets around $1 + x^3$, whereas there are modulus signs in the given answer. The Mathcad answer is thus valid only for a restricted set of values of x, namely, $1 + x^3 > 0$.

Solution 5.6

You should have obtained the following Mathcad answers, in which the numerical values are given to 3 decimal places.

(a) $\dfrac{2}{\pi} = 0.637$

(b) 2

(c) $\frac{3}{4}\exp(4) - \frac{1}{4}\exp(2) = 39.101$

(d) $\frac{1}{9} - \frac{1}{18}2^{1/2} + \frac{1}{72}\pi 2^{1/2} = 0.094$

The exact answers given by Mathcad agree with those in the main text, though there is some algebraic rearrangement. The numerical answers given by Mathcad also agree with those in the main text, at least to the number of decimal places that the answers have in common. To increase the number of decimal places given by Mathcad, it is necessary to change the value of 'Number of decimal places', as described in the first Mathcad note for Activity 5.5, on page 16 of this computer book.

Solution 5.8

(a) 0.869

(b) 0.566

(c) 1.571

Chapter C3

Solution 5.3

(a) Up to the term in x^5, the Taylor series about 0 is

$$2 + 3x + 4x^2 + 5x^3 + 6x^4 + 7x^5 + \cdots.$$

The part of the series consisting of the next three terms is

$$\cdots + 8x^6 + 9x^7 + 10x^8 + \cdots.$$

In each term of the series the coefficient is two more than the power of x, so the whole series is

$$\sum_{k=0}^{\infty}(k+2)x^k.$$

(b) Up to the term in x^5, the Taylor series about 0 is

$$-x + 4x^2 - 9x^3 + 16x^4 - 25x^5 + \cdots.$$

The part of the series consisting of the next three terms is

$$\cdots + 36x^6 - 49x^7 + 64x^8 + \cdots.$$

In each term of the series the coefficient is the square of the power of x, multiplied by -1 when the power is odd, so the whole series is

$$\sum_{k=0}^{\infty}(-1)^k k^2 x^k.$$

(The lower limit of the sum here could be taken to be 1 instead of 0, since the term obtained by taking $k = 0$ is 0.)

Solution 5.6

(a) The function is neither even nor odd. Reasonable values for the x- and y-axis limits in this case are $X1 = -5$, $X2 = 5$, $Y1 = -5$, $Y2 = 5$. It appears that the maximum range of validity is $-3 < x < 3$.

(b) The function is odd. Reasonable values for the x- and y-axis limits in this case are $X1 = -2$, $X2 = 2$, $Y1 = -1$, $Y2 = 1$. It appears that the maximum range of validity is $-\frac{1}{2} < x < \frac{1}{2}$.

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